Boundary Value Problems As Sturm-Liouville Problems

The following is a **Sturm-Liouville** problem:

ODE $X''(x) + \lambda X(x) = 0 \quad 0 < x < 1$ -----------(1)

BCs

$X(0) = 0$ ------------(2)

$X'(1) = 0$ -------------(3)

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where ' implies first derivative and " implies second derivative.

(a) What is a **Sturm-Liouville** problem?

(b) What are the **eigenvalues** (also called **characteristic values**) and **eigenfunctions** (also called **characteristic functions**) of the given Sturm-Louiville problem?

Solution

(a) A **Sturm-Liouville** problem is a *boundary-value problem* and its general form is as follows:

ODE $[p(x)y']' - q(x)y + \lambda r(x)y = 0 \quad 0 < x < 1$ --------------(4)

BCs

$a_1 y(0) + b_1 y'(0) = 0$ ---------------(5)

$a_2 y(1) + b_2 y'(1) = 0$ ---------------(6)

Sturm and Liouville showed that the **ODE in X(x)** and its associated **BCs** (derived when the **separation of variables** method is used to solve **PDEs** with *linear homogeneous boundary conditions*) will always be some particular Sturm-Liouville problem.

(b) The general form of the solution of equation (1) is as follows:

$$X(x) = A \sin(\lambda^{1/2} x) + B \cos(\lambda^{1/2} x) \quad \text{-----------------------------(7)}$$

Equation (7) must now satisfy the BCs given in equations (2) and (3).

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So, $X(0) = A \sin(\lambda^{1/2}0) + B \cos(\lambda^{1/2}0) = 0 \Rightarrow B = 0$

and $X'(1) = A \cos(\lambda^{1/2}1) - B \sin(\lambda^{1/2}1) = 0 \Rightarrow \cos(\lambda^{1/2}) = 0$

Therefore, $\lambda_n^{1/2} = n\,\pi/2$ $\quad n = 1, 3, 5, \ldots$

So $\lambda_n = (n\,\pi/2)^2$ $\quad n = 1, 3, 5, \ldots$

So, **eigenvalues =** $\boldsymbol{\lambda_n = (n\,\pi/2)^2}$ $\quad n = 1, 3, 5, \ldots$

and **eigenfunctions =** $\boldsymbol{X_n(x) = \sin(n\pi x/2)}$ $\quad n = 1, 3, 5, \ldots$

The String: $S_7P_2A_{21}$ (Identity – Physical Properties).

The Pj Problem of interest is of type i*dentity*. All problems of mathematical modeling are identity problems because the problems seek the mathematical structure of the physical problem being modeled.