Eigenfunction Expansion Solution Method For Non-Homogeneous PDEs

The following *initial boundary value problem* (IBVP) is for heat flow in a one-dimensional unit rod. IBVP is non-homogeneous because $f(x,t) \neq 0$:

PDE	$u_t = \alpha^2 u_{xx} + f(x,t)$	$0 < x < 1$	$0 < t < \infty$
BCs	$u(0,t) = 0$ $u(1,t) = 0$	$0 < t < \infty$	
IC	$u(x,0) = g(x)$	$0 \leq x \leq 1$	

Determine the function **u (x,t)** by the **eigenfunction expansion method**

where **u(x,t)** represents the temperature at some point **x** along the rod and at some point in time, **t**.

$u_t = \delta u/\delta t$; $u_{xx} = \delta^2 u/\delta x^2$

α^2 = diffusivity (cm^2/sec)

f(x,t) = some heat source within rod

PDE (partial differential equation)

BCs (boundary conditions)

IC (initial condition).

Assertion 1: the difference between the given non-homogeneous IBVP and its associated homogeneous IBVP is the heat source **f(x,t)**.

Assertion 2: **f(x,t)** can be decomposed into simple components such that:

$$f(x,t) = f_1(t)X_1(x) + f_2(t)X_2(x) + \ldots + f_n(t)X_n(x) \text{ ---------(1)}$$

where $X_n(x)$ are the **eigenfunctions** of the homogeneous IBVP associated with the non-homogeneous IBVP.

Assertion 3: each simple individual component has a response $u_n(x,t)$.

Assertion 4: **u(x,t)** is the infinite sum of the responses $u_n(x,t)$, that is:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) \text{ --(2)}$$

Eigenfunction Expansion Solution Method For Non-Homogeneous PDEs

Decomposing f(x,t)

The Sturm-Liouville problem of the homogeneous IBVP associated with the given non-homogeneous IBVP is as follows:

$$X''(x) + \lambda^2 X(x) = 0 \text{ -----------(3)}$$

$$X(0) = 0 \text{ -----------(4)}$$

$$X(1) = 0 \text{ ----------(5)}$$

So, the eigenfunctions $X_n(x) = \sin(n\pi x)$ $n = 1, 2, \dots,$

So, $f(x,t) = f_1(t)\sin(\pi x) + f_2(t)\sin(2\pi x) + \dots + f_n(t)\sin(n\pi x)$-------------(6)

In order to find the function $f_n(t)$, equation (6) is multiplied by $\sin(m\pi x)$ and integrated with respect to x as follows:

So, $$\int_0^1 f(x,t)\sin(m\pi x)\,dx = \sum_{n=1}^{\infty} f_n(t)\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx \text{------------(7)}$$

$$= f_m(t)/2$$

So, $$f_n(t) = 2\int_0^1 f(x,t)\sin(n\pi x)\,dx \text{---------------------------(8)}$$

So, $$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x) \text{ ----------------------------(9)}$$

Now, the response to each individual input $f_n(t)X_n(x)$, is

$$u_n(x,t) = T_n(t)X_n(x)$$

So, $$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x) = \sum T_n(t)\sin(n\pi x) \text{-------------(10)}$$

Eigenfunction Expansion Solution Method For Non-Homogeneous PDEs

Now, substituting,

$$\mathbf{u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)}$$

and

$$\mathbf{f(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x)}$$

in the given non-homogeneous IBVP , we have:

$$\mathbf{u_t = \sum_{n=1}^{\infty} T'_n(t)\sin(n\pi x)}$$

$$\mathbf{u_{xx} = -\alpha^2 \sum_{n=1}^{\infty} (n\pi)^2 T_n(t)\sin(n\pi x)}$$

So,

$$\mathbf{\sum_{n=1}^{\infty} T'_n(t)\sin(n\pi x) = -\alpha^2 \sum_{n=1}^{\infty} (n\pi)^2 T_n(t)\sin(n\pi x) + \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x)} \text{ --------------(11)}$$

$$u(0,t) = \mathbf{\sum_{n=1}^{\infty} T_n(t)\sin(0) = 0} \text{ ----------------------(12)}$$

$$u(1,t) = \mathbf{\sum_{n=1}^{\infty} T_n(t)\sin(n\pi) = 0} \text{ ---------------------(13)}$$

$$u(x,0) = \mathbf{\sum_{n=1}^{\infty} T_n(0)\sin(n\pi x) = g(x)} \text{ ----------------(14)}$$

Eigenfunction Expansion Solution Method For Non-Homogeneous PDEs

Equations (12) and (13) are trivial, that is, zero = zero. So we are left with equations (11) and (14) both of which constitute an initial value problem. The PDE expressed in equation (11) and the IC expressed in equation (14) can be rewritten as follows:

PDE
$$\sum_{n=1}^{\infty} \left[T'_n(t) + (n\alpha\pi)^2 T_n(t) - f_n(t)\right] \sin(n\pi x) = 0$$

So,
$$\sum_{n=1}^{\infty} \left[T'_n(t) + (n\alpha\pi)^2 T_n(t) - f_n(t)\right] = 0 \text{ ---------------------(15)}$$

IC
$$\sum_{n=1}^{\infty} T_n(0) \sin(n\pi x) = g(x) \text{ ---------------------(16)}$$

So, $T_n(t)$ satisfies the following ODE:

$$T'_n + (n\alpha\pi)^2 T_n = f_n(t) \text{ ---------------------(17)}$$

$$T_n(0) = 2\int_0^1 g(\xi) \sin(n\pi\xi)\, d\xi = a_n \text{---------------------(18)}$$

By using an integrating factor, we have the solution of the ODE (17 and 18) as follows:

$$T_n(t) = a_n e^{-(n\pi\alpha)^2 t} + \int_0^t e^{-(n\pi\alpha)^2 (t-\tau)} f_n(\tau)\, d\tau \text{-------(19)}$$

So,
$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(n\pi x)$$

$$= \sum_{n=1}^{\infty} \left[a_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x)\right] + \sum_{n=1}^{\infty} \left[\sin(n\pi x) \int_0^t e^{-(n\pi\alpha)^2 (t-\tau)} f_n(\tau)\, d\tau\right] \text{--------------(20)}$$

Eigenfunction Expansion Solution Method For Non-Homogeneous PDEs

So, solution of given non-homogeneous IBVP is:

$$u(x,t) = \sum_{n=1}^{\infty}[a_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x)] + \sum_{n=1}^{\infty}[\sin(n\pi x)\int_0^t e^{-(n\pi\alpha)^2(t-\tau)} f_n(\tau)\, d\tau] \text{---------}(20)$$

Transient part + Steady state

The transient part is due to the initial condition while the steady state part is due to f(x,t).

The String: $S_7P_2A_{21}$ (Identity – Physical Properties).

The Pj Problem of interest is of type i*dentity*. All problems of mathematical modeling are identity problems because the problems seek the mathematical structure of the physical problem being modeled.