Homogenizing Non-homogeneous Time Varying Boundary Conditions

The solution of PDEs with the separation of variables method is only possible when the IBVP problem is linearly homogeneous. When the boundary conditions (BCs) are non-homogeneous, it is often desirable to transform them to homogeneous BCs.

Consider the following IBVP for a one -dimensional heat flow in a laterally insulated rod of length L:

PDE	$u_t = \alpha^2 u_{xx}$	$0 < x < L$	$0 < t < \infty$
BCs	$u(0,t) = g_1(t)$ $u_x(L,t) + hu(L,t) = g_2(t)$	$0 < t < \infty$	non-homogeneous BCs
IC	$u(x,0) = p(x)$	$0 \le x \le L$	

Transform the **non-homogeneous BCs** to **homogeneous BCs**.

where **u(x,t)** represents the temperature at some point **x** along the rod and at some point in time, **t**.
$u_t = \delta u/\delta t$; $u_{xx} = \delta^2 u/\delta x^2$; **h =** heat exchange coefficient.
α^2 = diffusivity (cm^2/sec)
PDE (partial differential equation)
BCs (boundary conditions)
IC (initial condition).

The objective is to express **u(x,t)** as a sum of two responses:
. The response due to the non-homogeneous BCs (**steady state response**)
. The response due to the IC (**transient response**).
Such that the steady state response satisfies the BCs.

The search for the steady state response is sometimes through trial and error and for the given IBVP it is:

$$S(x,t) = A(t)\,[1 - x/L] + B(t)[x/L] \qquad \text{------------------------(1)}$$

So, $$u(x,t) = S(x,t) + U(x,t) \qquad \text{------------------------(2)}$$

where S(x,t) is the steady state response and U(x,t) is the transient response.

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If S(x,t) must satisfy the BCs, then:

$$\mathbf{S(0,t) = A(t) = g_1(t)}$$

$$\mathbf{S_x(L,t) + hS(L,t) = -A(t)/L + B(t)/L + hB(t) = g_2(t)}$$

So,

$$\mathbf{B(t) = [g_1(t) + Lg_2(t)]/(1 + Lh)}$$

So,

$$\mathbf{u(x,t) = g_1(t)[1 - x/L] + [(g_1(t) + Lg_2(t))](1 + Lh)](x/L) + U(x,t)} \text{ -----------}(3)$$

By substituting equation (3) into the given IBVP with non-homogeneous BCs, we get the following transformed IBVP:

PDE $\mathbf{U_t = \alpha^2 U_{xx} - S_t}$ $\quad 0 < x < L \quad 0 < t < \infty$

BCs $\mathbf{U(0,t) = 0}$, $\mathbf{U_x(L,t) + huU(L,t) = 0}$ $\quad 0 < t < \infty \quad$ homogeneous BCs

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IC $\mathbf{U(x,0) = p(x) - S(x,0)}$ $\quad 0 \le x \le L$

The BCs have been homogenized. However, the PDE became non-homogeneous as a result. So, the separation of variables method cannot be used to solve this transformed IBVP. Non-homogeneous PDEs are solved with **eigenfunction expansions** and **integral transform** methods. The Laplace transform can solve non-homogeneous PDEs without requiring that the BCs be homogenized.

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The String: $S_7P_2A_{21}$ (Identity – Physical Properties).

The Pj Problem of interest is of type i*dentity*. All problems of mathematical modeling are identity problems because the problems seek the mathematical structure of the physical problem being modeled.