Infinite Sequences And Series

/ is the division symbol

An ordered set of objects is called a **sequence**. A sequence can also be defined as a function whose domain is the set of positive integers. The sequence is **infinite** if for any n^{th} term of the sequence there always exists a $(n+1)^{th}$ term.

(1) Consider the sequence $a_n = n/(2n + 1)$.
What is the 7^{th} term of the sequence? Does the sequence converge to a limit? If so what is that limit?

Ans (1) 7^{th} term = f(7) = 7/(14 + 1) = 7/15
$\lim_{n \to \infty} n/(2n + 1) = \lim_{n \to \infty} 1/(2 + 1/n) = ½$. Sequence converges and its limit is ½.

(2) Is the infinite sequence $\lim_{n \to \infty} 1/5^n$ convergent?

Ans (2) $\lim_{n \to \infty} 1/5^n = \lim_{n \to \infty} (1/5)^n = 0$. Sequence is convergent.
This sequence belongs to the group of sequences called **geometric progression** (G.P.) with common ratio r, where r = $(n + 1)^{th}$ term/n^{th} term and |r| < 1.

The sum of the terms of a sequence is called a **series**. The sum of an infinite sequence is called an **infinite series**. Consider the series:

$\sum_{n=1}^{\infty} a_n$. The partial sums are :

$S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$
.
.
.
$S_n = a_1 + a_2 + a_3 + \dots + a_n$
.
.
.

$\lim_{n \to \infty} S_n = S$ implies S is the sum of the infinite series. The series is said to converge to S.

(3) What is the sum of $\lim_{n \to \infty} 1/5^n$?

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Ans (3) In general, sum of first n terms of a G. P., $S_n = a(1 - r^n)/(1 - r)$. $\lim_{n \to \infty} S_n = a/(1 - r) = 1/4$.

Power Series are of the form: $\sum_{n=0}^{\infty} a_n x^n$. Where x is a variable and the coefficients of x are constants.

A power series of the form: $\sum_{n=0}^{\infty} a_n(x-a)^n$ is a power series centered at a or about a.

A power series in x implies a function f(x) over the domain of values of x for which the series converges. This domain is called the i**nterval of convergence**.

The **radius of convergence** of a power series about a is:

(a) 0 if the series converges only when $x = a$

(b) ∞ if the series converges for all x

(c) A positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

The **interval of convergence** for a radius of convergence = 0 is a.

The **interval of convergence** for a radius of convergence = ∞ is all real numbers $(-\infty, \infty)$

The **interval of convergence** for a radius of convergence = R is either $[a-R, a+R]$, or $(a-R, a+R]$, or $[a-R, a+R)$, or $(a-R, a+R)$.

Convergence Tests

D'Alembert's Ratio Test: $u_k > 0$; $\lim_{k\to\infty} u_{k+1}/u_k < 1$ implies $\sum_{k=0}^{\infty} u_k$ converges.

Comparison Test: $0 \le u_k \le v_k$; $\sum_{k=0}^{\infty} v_k$ converges implies $\sum_{k=0}^{\infty} u_k$ converges.

Leibniz's Alternating Series Test: $v_k \downarrow$; $v_k \ge 0$; $\lim_{k\to\infty} v_k = 0$ implies $\sum_{k=0}^{\infty} (-1)^k v_k$ converges.

Absolute Convergence Test: $\sum_{k=1}^{\infty} u_k$ converges absolutely if and only if $\sum_{k=1}^{\infty} |u_k|$ converges

Conditional Convergence Test: $\sum_{k=1}^{\infty} u_k$ converges conditionally if and only if $\sum_{k=1}^{\infty} |u_k|$ diverges.

Cauchy's Test: $u_k > 0$, k= 1,2,... ; $\lim_{k\to\infty} (u_k)^{1/k} < 1$ implies $\sum_{k=1}^{\infty} u_k$ converges

MaClaurin's Test: $f(x) \ge 0$; $\in C$, $\downarrow$ (means f(x) is nonincreasing)

$$\lim_{R\to\infty} \int_1^R f(x)dx = A; \text{ implies } \sum_{k=1}^{\infty} f(k) \text{ converges.}$$

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(4) Given the power series $\sum_{n=0}^{\infty} x^n/(n + 2)$

Determine its radius of convergence if it converges and the interval of convergence.

Ans (4) By the ratio test, $\lim_{n\to\infty} | (x^{n+1}/(n+3))((n+2)/x^n) | = |x| \lim_{n\to\infty} (n+2)/(n+3) = |x|$

So, $|x| < 1$ for convergence. So radius of convergence = 1

If x =1, series becomes $\sum_{n=0}^{\infty} 1/(n+2)$. This series diverges by the comparison test with the harmonic series.

The harmonic series is: $\sum_{n=0}^{\infty} 1/n$

If x = -1, series becomes $\sum_{n=0}^{\infty} (-1)^n/(n+2)$. This series converges by the alternating series test.

So, interval of convergence = [-1,1).

(5) Given the power series $\sum_{n=0}^{\infty} x^n/n!$

Determine the radius of convergence if it converges and the interval of convergence.

Ans (5) By the ratio test, $\lim_{n\to\infty} | (x^{n+1}/(n+1)!)/(x^n/n!) | = |x| \lim_{n\to\infty} 1/(n+1) = 0 < 1$ for all x.

So, radius of convergence = ∞; Interval of convergence = (-∞, ∞).

(6) Given the power series $\sum_{n=0}^{\infty} n!(2x-1)^n$

Determine its radius of convergence if it converges and the interval of convergence.

Ans (6) By the ratio test, $\lim_{n\to\infty} | [(n+1)!(2x-1)^{n+1}]/[n!(2x-1)^n] | = \lim_{n\to\infty} | (n+1)(2x-1) | \to\infty$ for all x ≠ ½.

So, since series diverges for all x ≠ ½, radius of convergence = 0 and interval of convergence = ½.

Representing functions by infinite series: instances exist whereby the best way to approximate a function is to represent it with an infinite series. Three infinite series are commonly used to represent functions when the functions meet certain criteria. These series are the **Maclaurin series**, the **Taylor series** and the **Binomial series**.

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Taylor Series: if a Taylor series centered at *a* and of n terms represent f(x), then the $(n+1)^{th}$ derivative of f(x) at *a* must exist. That is, $f(x) \in C^{n+1}$. The representation is expressed as:

$$f(x) = \sum_{n=0}^{\infty} (f^{\mathbf{n}}(a) / n!)(x-a)^n.$$

The **Maclaurin series** is the Taylor series centered at 0. So representation is expressed as:

$$f(x) = \sum_{n=0}^{\infty} (f^n(0) / n!)x^n.$$

(7) Represent $f(x) = e^x$ by its Maclaurin series.

Ans (7) $df(x)/dx = e^x$ for all n derivatives. So,$f^n(0) =1$ for all n. So, Maclaurin series for $f(x) = e^x$ is:

$$\text{So, } f(x) = \sum_{n=0}^{\infty} (f^n(0) / n!)x^n = 1 + x + x^2/2! \ldots x^n/n! \ldots, \text{ all } x$$

The Maclaurin series for some other commonly used functions are:

$\sin x = x - x^3/3! + x^5/5! + \ldots + (-1)^{n+1}[x^{2n-1}/(2n-1)!] + \ldots$, all x

$\cos x = 1 - x^2/2! + x^4/4! - \ldots + (-1)^{n+1}[x^{2n-2}/(2n-2)!] + \ldots$, all x

$\tan x = x + x^3/3 + 2x^5/15 + 17x^7/315 + \ldots$, for $x^2 < \pi^2/4$

$\ln (1 + x) = x - x^2/2 + x^3/3 - \ldots + (-1)^{n+1}(x^n/n) + \ldots$, for $-1 < x \leq 1$

Binomial series: $f(x) = (1 + x)^k = \sum_{n=0}^{\infty} [k!/((k-n)!n!)]x^n$

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Simate was my mother
Sagay was my father