Probability Density Functions

/ is the division symbol

Identity: $f(x) \geq 0$ in domain of X and $\int_{-\infty}^{\infty} f(x)\,dx = 1$; implies f(x) is a probability density function.

The identity of a probability function implies that the graph of f(x) must lie above or on the x axis and the area under the graph must be equal to 1 for all values in the domain of X.

(1) If $f(x) = kx^2$, determine the value of k that makes f(x) a probability density function on $0 \leq x \leq 2$.

Ans (1) $\int_0^2 kx^2\,dx = k\,[x^3/3]_0^2 = 8k/3 = 1$ implies $k = 3/8$. Clearly $f(x) \geq 0$ on $0 \leq x \leq 2$.

(2) $f(x) = 6(x - x^2)$ is a probability function on $0 \leq x \leq 1$. Determine the following probabilities
(a) $1/4 \leq x \leq 1/2$ (b) $x \geq 1/4$ (x at least ¼) (c) $x \leq 3/4$ (x at most ¾)

Ans (2a) $\int_{1/4}^{1/2} 6(x - x^2)dx = 6[x^2/2 - x^3/3]_{1/4}^{1/2} = ½ - 5/32 = 11/32.$

(b) $6[x^2/2 - x^3/3]_{1/4}^{1} = 27/32.$

(c) $6[x^2/2 - x^3/3]_{0}^{3/4}$

Constant value probability density function: $f(x) = 1/(B - A)$; $A \leq x \leq B$

(3) An automated machine produces an automobile part every 3 minutes. An inspector arrives at random time and must wait X minutes for a part.
(a) Find the probability density function for X
(b) Find the probability that the inspector must wait at least 1 minute.
(c) Find the probability that the inspector must wait no more than 1 minute

Ans (3a) $f(x) = 1/3$; $0 \leq x \leq 3$

(b) $\int_1^3 1/3\,dx = 1/3\,[x]_1^3 = 2/3$

(c) $1/3\,[x]_0^1 = 1/3$

Exponential probability density function: $f(x) = \lambda e^{-\lambda}$. a = aversge value of X, implies $\lambda = 1/a$.

(4) The service time X at a gas station has an exponential probability density function. It takes an average of 4 minutes to get serviced.
(a) What fraction of the cars are serviced within 2 minutes?
(b) What is the probability that a car will have to wait at least 4 minutes?

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Ans (4a) average value a = 4, so $\lambda = 1/a = 1/4$

So, $f(x) = 1/4e^{-x/4}$ $x \geq 0$; So, $\int_0^2 1/4e^{-x/4}\,dx = [-e^{-x/4}]_0^2 = 0.3934$.

(b) $[-e^{-x/4}]_4^{\infty} = -0 + e^{-1} = 0.36788$.

(5) The exponential probability density function is used to model the inter-arrival times in seconds between successive cars at a given toll booth. Find the probability that X is at least 3 seconds, if the average inter-arrival time is 2 seconds.

Ans (5) $\int_3^{\infty} 1/2e^{-x/2}\,dx = [-e^{-x/2}]_3^{\infty} = 0.22313$.

(6) The exponential probability density function with an average value of 2 years is used to model the elapsed time between successive retirements of Justices of the U. S Supreme Court. Suppose a new President takes office at the same time a Justice retires. What is the probability that the next vacancy will take place during his 4-year term?

Ans (6) $\int_0^4 1/2e^{-x/2}\,dx = [-e^{-x/2}]_0^4 = 1 - e^{-2} = 1 - 0.13534 = 0.86466$.

(7) The probability density function of the survival times X, after a group of patients have been treated for a given ailment is $f(x) = ke^{-kx}$ for some constant k. The survival function S(x) is the probability that a person chosen at random from the group of patients survives at least x years. Suppose S(5) = 0.90 find the constant k in f(x).

Ans (7) $\int_5^{\infty} ke^{-kx}\,dx = [-e^{-kx}]_5^{\infty} = e^{-5k} = 0.90$. So, $-5k = \ln 0.90$ from which $k = 0.0211$.

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Simate was my mother
Sagay was my father.