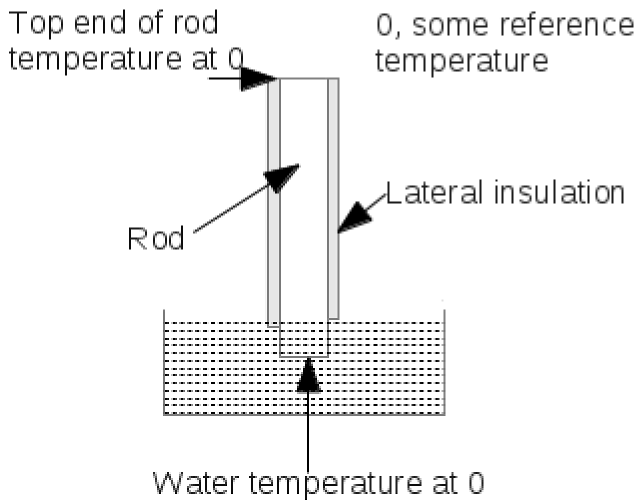

Figure 14.14

Figure 14.14 shows a one-dimensional heat flow problem. The bottom end of a laterally insulated unit rod is immersed in a water solution at a fixed reference temperature. The temperature at the top end is also fixed at the same reference temperature. The *initial boundary value problem* (IBVP) of the heat flow problem is as follows:

PDE $u_t = \alpha^2 u_{xx}$ $\quad 0 < x < 1 \quad 0 < t < \infty$

BCs

$u(0,t) = 0$

$u_x(1,t) + hu(1,t) = 0$ $\quad 0 < t < \infty$ homogeneous BCs

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IC $u(x,0) = x$ $\quad 0 \le x \le 1$

Determine the function **u (x,t)** by the **separation of variables method**.

where **u(x,t)** represents the temperature at some point **x** along the rod and at some point in time, **t**.

$u_t = \delta u/\delta t$; $\quad u_{xx} = \delta^2 u/\delta x^2$; **h =** heat exchange coefficient.

α^2 = diffusivity (cm^2/sec)

PDE (partial differential equation)

BCs (boundary conditions)

IC (initial condition).

In Search Of The Function u (x,t) By The Separation Of Variables Method.

Assertion 1: there exists functions $X_n(x)$ and $T_n(t)$ such that:

$$\mathbf{u_n (x,t) = X_n(x) T_n(t)} \text{ --------(1)}$$ (called **fundamental solutions**)

Assertion 2: the identity of **u(x,t)** is the same as the identity of the infinite sum of **u_n (x,t)** that satisfies the given **IBVP**. That is:

$$\mathbf{u (x,t) = \sum_{n=1}^{\infty} A_n u_n (x,t) = \sum_{n=1}^{\infty} A_n X_n(x) T_n(t)} \text{-------(2)}$$ (if IBVP is satisfied)

Separating Variables:

$$\mathbf{u (x,t) = X(x)T(t) \text{ and } u_t = \alpha^2 u_{xx}}$$

Implies:

$$\mathbf{X(x)T'(t) = \alpha^2 X''(x)T(t)} \text{-------(3)}$$

where $T'(t) = \delta T/\delta t = u_t = \delta u/\delta t$; and $X'' = \delta^2 X/\delta x^2 = u_{xx} = \delta^2 u/\delta x^2$

Dividing equation (3) by **$\alpha^2 X(x)T(t)$**, we have :

$$\mathbf{X(x)T'(t)/ \alpha^2 X(x)T(t) = \alpha^2 X''(x)T(t)/ \alpha^2 X(x)T(t)}$$

So, $$\mathbf{T'(t)/\alpha^2 T(t) = X''(x)/ X(x)} \text{---------(4)}$$

The left hand side of equation (4) depends only on t and the right hand side depends only on x. Since x and t are independent, equation (4) implies that:

$$\mathbf{T'(t)/\alpha^2 T(t) = \mu}$$ (where μ is the separation constant)

and

$$\mathbf{X''(x)/ X(x) = \mu}$$

So, $$\mathbf{T'(t) - \mu \alpha^2 T(t) = 0} \text{----------(5)}$$

and

$$\mathbf{X''(x) - \mu X(x) = 0} \text{---------(6)}$$

Equations (5) and (6) separate the variables and reduce the PDE to two ODEs.

$$\mathbf{T'(t) - \mu\, \alpha^2 T(t) = 0} \text{----------(5)}$$

$$\mathbf{X''(x) - \mu\, X(x) = 0} \text{----------(6)}$$

$\mu < 0$ is the domain of μ for which equations (5) and (6) are meaningful.
If $\mu > 0$, $u(x,t) = X(x)T(t)$ tends to infinity. If $\mu = 0$, $u(x,t) = 0$.
μ is set equal to $-\lambda^2$ for $\mu < 0$. So, equations (5) and (6) become:

$$\mathbf{T'(t) + \lambda^2 \alpha^2 T(t) = 0} \text{----------(7)}$$

$$\mathbf{X''(x) + \lambda^2 X(x) = 0} \text{----------(8)}$$

The solutions for equations (7) and (8) are as follows:

$$\mathbf{T(t) = Ae^{-(\lambda\alpha)^2 t}} \text{ ---------(9)}$$

$$\mathbf{X(x) = B \sin(\lambda x) + C \cos(\lambda x)} \text{---------(10)}$$

So, $$\mathbf{u(x,t) = X(x)T(t) = e^{-(\lambda\alpha)^2 t}[A \sin(\lambda x) + B \cos(\lambda x)]} \text{-----------(11)}$$

satisfies the PDE, $u_t = \alpha^2 u_{xx}$ $\quad 0 < x < 1 \quad\quad 0 < t < \infty$

for any λ and any A and B.

There are infinitely many **u (x,t),** as expressed in equation (11) that satisfy the PDE.
We now look for those that satisfy both the PDE and the boundary conditions (BCs):

$$\mathbf{u(0,t) = 0}$$
$$\mathbf{u_x(1,t) + hu(1,t) = 0}$$

So, substituting $e^{-(\lambda\alpha)^2 t}[A \sin(\lambda x) + B \cos(\lambda x)]$ into the BCs, we have:

$$\mathbf{Be^{-(\lambda\alpha)^2 t} = 0 \Rightarrow B = 0}$$
$$\mathbf{A\lambda e^{-(\lambda\alpha)^2 t} \cos\lambda + hAe^{-(\lambda\alpha)^2 t} \sin\lambda = 0}$$

So, $$\mathbf{\tan\lambda = -\lambda/h} \text{ -------(12)}$$

$$\tan \lambda = -\lambda/h \text{ -------(12)}$$

The values of **λ** for a given value of **h** (can be computed numerically with the aid of a computer) for which equation (12) is satisfied are called the **eigenvalues** of the boundary-value problem:

$$X''(x) + \lambda^2 X(x) = 0 \text{ -----------(13)}$$

$$X(0) = 0 \text{ -----------(14)}$$

$$X(1) + hX(1) = 0 \text{ -----------(15)}$$

These **eigenvalues** are the values of λ for which there exists a *nonzero solution* for the boundary-value problem. The solutions of the boundary-value problem derived from the eigenvalues λ_n are called the **eigenfunctions**, $X_n(x)$. For this boundary-value problem (equations 13 thru 15):

$$X_n(x) = \sin(\lambda_n x)$$

So, the infinite number of **fundamental functions** can be expressed as follows:

$$u_n(x,t) = X_n(x)\, T_n(t) = e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x) \text{ ---------------(16)}$$

Each of these functions satisfy the PDE and the BCs. Their sum such that the initial condition IC is satisfied is the identity of u (x,t).

So, $$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x) \text{ ------------------------(17)}$$

such that the initial condition (IC) , u(x, 0) = 0 is satisfied. That is:

$$u(x,0) = x = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x) \text{ ------------------------------(18)}$$

$$u(x, 0) = x = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x) \text{ ------------------------------(18)}$$

The constants $\mathbf{a_n}$ in the eigenfunction expansion (equation 18) can be determined by multiplying each side of equation (18) by $\mathbf{\sin(\lambda_m x)}$ and integrating x from 0 to 1:

$$\int_0^1 x \sin(\lambda_m x)\, dx = \sum_{n=1}^{\infty} a_n \int_0^1 \sin(\lambda_n x) \sin(\lambda_m x)\, dx \text{------------(19)}$$

let x = ξ ; then, dx/dξ = 1

So, equation (19) becomes:

$$\int_0^1 \xi \sin(\lambda_m \xi)\, d\xi = \sum_{n=1}^{\infty} a_n \int_0^1 \sin(\lambda_n \xi) \sin(\lambda_m \xi)\, d\xi \text{------------(20)}$$

$$= a_m \int_0^1 \sin^2(\lambda_m \xi)\, d\xi$$

$$= a_m (\lambda_m - \sin\lambda_m \cos\lambda_m)/2\lambda_m$$

So, $$a_m = 2\lambda_m/(\lambda_m - \sin\lambda_m \cos\lambda_m) \int_0^1 \xi \sin(\lambda_m \xi)\, d\xi$$

Changing notation to *n*, we have:

$$a_n = 2\lambda_n/(\lambda_n - \sin\lambda_n \cos\lambda_n) \int_0^1 \xi \sin(\lambda_n \xi)\, d\xi \text{-----(21)}$$

So, the solution to the IBVP problem is:

So, $$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x) \text{ ------------------------(22)}$$

where the constants $\mathbf{a_n}$ are calculated from equation (21). Separation method is valid only for homogeneous IBVP. Other methods are used to solve non-homogeneous IBVP.

The String: $S_7P_2A_{21}$ (Identity – Physical Properties).

The Pj Problem of interest is of type i*dentity*. All problems of mathematical modeling are identity problems because the problems seek the mathematical structure of the physical problem being modeled.