Advanced Calculus – Jacobians

/ is the division symbol

The mathematical modeling of complex existential problems often result in systems of partial differential equations. Several methods exist for determining the partial derivatives of such equations. The **Jacobian** is an effective method used to determine the partial derivatives of a system of partial differential equations, particularly when the number of equations are greater than two.

Consider the scenario with the following propositions:
(a) F(u, v, w, x, y, z), G(u, v, w, x, y, z), H(u, v, w, x, y, z), f(x, y, z), g(x, y, z), h(x, y, z) ϵ C^1
(b) F(f(x, y, z), g(x, y, z), h(x, y, z), x, y, z) ≡ 0
(c) G(f(x, y, z), g(x, y, z), h(x, y, z), x, y, z) ≡ 0
(d) H(f(x, y, z), g(x, y, z), h(x, y, z), x, y, z) ≡ 0
Then there exist, the **Jacobians** of F, G, H with respect to any subset of the variables of F, G, and H. For example, the Jacobian of F, G, H with respect to u, v, x is as follows:

$$\boldsymbol{\delta (F, G, H) / \delta (u, v, x)} = \begin{vmatrix} \mathbf{F_1} & \mathbf{G_1} & \mathbf{H_1} \\ \mathbf{F_2} & \mathbf{G_2} & \mathbf{H_2} \\ \mathbf{F_4} & \mathbf{G_4} & \mathbf{H_4} \end{vmatrix}$$

Note how positions of the variables in F, G, and H is reflected in the partial derivatives in determinant **.**

And, **δu / δx = - [δ (F, G, H) / δ (u, v, x)] / [δ (F, G, H) / δ (u, v, w)];** δ (F, G, H) / δ (u, v, w) ≠ 0.

Other partial derivatives implicit in F, G, H can be similarly determined via their respective Jacobians.

(1) Determine δ (F, G, H) / δ (u, w, v), If:
F = xu + v – y
G = u^2 + vy + w
H = zu – v + vw

Ans (1)

$$\delta (F, G, H) / \delta (u, w, v) = \begin{vmatrix} F_1 & G_1 & H_1 \\ F_3 & G_3 & H_3 \\ F_2 & G_2 & H_2 \end{vmatrix} = \begin{vmatrix} x & 2u & z \\ 0 & 1 & v \\ 1 & y & w-1 \end{vmatrix} = x(w-1-yv) -2u(-v) + z(-1)$$

= xw -x -xyv +2uv - z = 2uv + xw -x – xyv – z.

(2) Find δu / δx, δu / δy by use of Jacobians If u = f(u, v, x) v = g(u, v, y)

Ans (2) Set F(u, v, x, y) = u – f(u,v, x) ; G(u, v, x, y) = v – g(u, v, y)
So, δu / δx = - [δ (F, G) / δ (x, v)] / [δ (F, G) / δ (u, v)] = $f_3(1-g_2) / [(1-f_1)(1- g_2) -f_2g_1]$
So, δu / δy = - [δ (F, G) / δ (y, v)] / [δ (F, G) / δ (u, v)] = $f_2g_3/ [(1-f_1)(1- g_2) -f_2g_1]$.

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(3) If $F(x, y, u) = 0$, is $\delta u / \delta x$ equal to the reciprocal of $\delta x / \delta u$?

Ans (3) $\delta u / \delta x = -(\delta F / \delta x)/ (\delta F / \delta u) = -F_1/F_3$
$\delta x / \delta u = -(\delta F / \delta u)/ (\delta F / \delta x = -F_3/F_1$
So, yes $\delta u / \delta x$ = reciprocal of $\delta x / \delta u$.

(4) If $F(x, y, u, v) = 0$, $G(x, y, u, v) = 0$, is $\delta u / \delta x$ equal to the reciprocal of $\delta x / \delta u$?

Ans (4) u, v as depedent variable:
$\delta u / \delta x = - (\delta(F , G)) / \delta(x, v))/ (\delta(F, G) / \delta(u,v)) = - (F_1G_4 – F_4G_1) / (F_3G_4 – F_4G_3)$
x, y as dependent variable:
$\delta x / \delta u = -(\delta(F , G))/ \delta(u, y))/ (\delta(F, G) / \delta(x, y)) = - (F_3G_2 – F_2G_3) / (F_1G_2 – F_2G_1)$
So, No $\delta u / \delta x \neq$ reciprocal of $\delta x / \delta u$.

(5) Find the derivative of u with respect to x if

$$xu + uv = u – x$$
$$v^2 + xv = u + x$$

Is the derivative total or partial ?

Ans (5) Set $F(u, v, x) = xu + uv – u + x = 0$
$G(u, v, x) = v^2 + xv - u + x = 0$
Then, $du/dx = -(\delta(F , G))/ \delta(x, v))/ (\delta(F, G) / \delta(u, v))$ $\quad (\delta(F, G) / \delta(u, v)) \neq 0$
$dv/dx = -(\delta(F , G))/ \delta(v, x))/ (\delta(F, G) / \delta(u, v))$
So, $du/dx = -(F_3G_2 – F_2G_3) / (F_1G_2 – F_2G_1)$
$= -[(u +1)(2v + x) – u(v + 1)] / [(x + v -1) (2v + x) – u(-1)]$.
Derivative is total since there is a single independent variable.

(6) If $F(u, v, g(u, v, x)) = 0$ and $G(u, v, h(u, v, x)) = 0$; Find $\delta u / \delta x$.

Ans (6) $\delta(F , G))/ \delta(x, v) = \begin{vmatrix} F_3g_3 & G_3h_3 \\ F_2 + F_3g_2 & G_2 + G_3h_2 \end{vmatrix} = [F_3g_3(G_2 + G_3h_2) – G_3h_3(F_2 + F_3g_2)]$

$\delta(F , G))/ \delta(u, v) = \begin{vmatrix} F_1 + F_3g_1 & G_1 + G_3h_1 \\ F_2 + F_3g_2 & G_2 + G_3h_2 \end{vmatrix} = [(F_1+F_3g_1)(G_2+G_3h_2) – (F_2+F_3g_2)(G_1+G_3h_1)] \neq 0$

So, $\delta u / \delta x = -(\delta(F , G))/ \delta(x, v)) / (\delta(F , G))/ \delta(u, v))$
$= -[F_3g_3(G_2 + G_3h_2) – G_3h_3(F_2 + F_3g_2)] / [(F_1+F_3g_1)(G_2+G_3h_2) – (F_2+F_3g_2)(G_1+G_3h_1)]$.

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Simate was my mother
Sagay was my father