Advanced Calculus - Partial Derivatives

/ is the division symbol

Partial derivatives exist because multivariable functions exist. Partial derivative is the differentiation of one of the independent variables of a multivariate function while the other indepent variables are held constant.

(1) f(x,y) = sinxy / *cos(x + y)*. Find δf / δx.

Ans (1): f(x, y) =u(x,y) /v(x,y) implies δf / δx = (v δu / δx - u δv / δx)/v^2 (Quotient Rule).
δu / δx = ycosxy. δv / δx = -sin(x + y)
So, δf / δx = [cos(x + y) ycosxy – sinxy(-sin(x + y))] / $\cos^2(x + y)$
So, δf / δx = [cos(x + y) ycosxy + sinxy(-sin(x + y))] / cos2(x + y) .

(2) f(x,y) = $\tan^2(x^2 – y^2)$. Find $f_1(x,y)$ and $f_2(1,2)$.

Ans (2): $f_1(x,y)$ = δf / δx = [2tan(x^2 – y^2)$\sec^2$(x^2 - y^2)]2x
$f_2(x,y)$ = δf / δy = [2tan(x^2 – y^2)$\sec^2$(x^2 - y^2)]-2y
So, $f_2(1,2)$ = 8 tan 3 $\sec^2$ 3 .

(3) If

u – v + 2w = x + 2z --------------(1)
2u + v – 2w = 2x – 2z -------------(2)
u – v + w = z – y-----------------(3)

Find : δu / δy, δv / δy , δw/ δy.

Ans (3) partial differentiation of (1) : δu / δy - δv / δy + 2δw / δy = 0 --------------(4)
partial differentiation of (2) : 2δu / δy + δv / δy - 2δw / δy = 0 -------------(5)
partial differentiation of (3) : δu / δy - δv / δy + δw / δy = -1 ---------------(6)
Add (4) and (5) to get δu / δy = 0
Add (5) and (6) to get δw / δy = 1
So, from (4) δv / δy = 2.

(4) If

$u^2 + x^2 + y^2 = 3$ ----------------(7)
$u – v^2 + 3x = 4$ ---------------(8)

Find : δu / δx, δu / δy , δv / δx, δv/ δy.

Ans (4) partial differentiation with respect to x of (7) : 2uδu / δx + 2x = 0 --------------(9)
partial differentiation with respect to x of (8) : δu / δx - 2v δv/ δx + 3 = 0 -----(10)
partial differentiation with respect to y of (7) : 2uδu / δy + 2y = 0 ----------------(11)
partial differentiation with respect to y of (8) : δu / δy - 2vδv / δy = 0 -----(12)
From (9), δu / δx = -x/u, u ≠ 0. So, from (10), δv / δx = -(x – 3u) / 2uv.
From (11), δu / δy = - y/u, u ≠ 0. So from (12), δv / δy = y / 2uv.

Advanced Calculus - Partial Derivatives

(5) $f(x,y) = x^{xy}$. Find $\delta f / \delta x$, $\delta f / \delta y$.

Ans (5) Let $f(x,y) = z = x^{xy}$.
So, $\log(z) = xy\log(x)$. So, $(1/z)\delta z / \delta x = (xy)/x + y\log x$.
So, $\delta f / \delta x = \delta z / \delta x = z(y + y\log x) = x^{xy}(y + y\log x)$.
$(1/z)\delta z / \delta y = 0 + x\log x$.
So, $\delta f / \delta y = \delta z / \delta y = z(x\log x) = x^{xy}(x\log x) = x^{xy+1}\log x$.

(6) If

$\log u + v = xy$ --------------(13)

$\log v + u = x - y$ -----------(14)

Find : $\delta u / \delta x$, $\delta v / \delta x$, $\delta v / \delta y$, $\delta v / \delta y$.

Ans (6) partial differentiation with respect to x of (13): $(1/u)\,\delta u / \delta x + \delta v / \delta x = y$ --------------(15)
partial differentiation with respect to x of (14): $(1/v)\,\delta v / \delta x + \delta u / \delta x = 1$ --------------(16)
partial differentiation with respect to y of (13): $(1/u)\,\delta u / \delta y + \delta v / \delta y = x$ --------------(17)
partial differentiation with respect to y of (14): $(1/v)\,\delta v / \delta y + \delta u / \delta y = -1$--------------(18)
Multiply (16) by 1/u then subtract (16) from (15):
So, $\delta v / \delta x(1 - 1/uv) = y - 1/u$, $u \neq 0$. So, $\delta v / \delta x = v(yu - 1)/(uv - 1)$.
Substitute $\delta v / \delta x$ in (15): So, $(1/u)\,\delta u / \delta x + v(yu - 1)/(uv - 1) = y$
So, $\delta u / \delta x = u(y - v) / (1 - uv)$.
Multiply (18) by 1/u then subtract (18) from (17):
So, $\delta v / \delta x(1 - 1/uv) = x + 1/u$ $u \neq 0$. So, $\delta v / \delta y = v(xu + 1) / (uv - 1)$.
Substitute $\delta v / \delta x$ in (17): So, $(1/u)\,\delta u / \delta y + v(xu + 1)/(uv - 1) = x$
So, $\delta u / \delta y = u(x + v) / (1 - uv)$.

(7) $u = e^v$ $v = \sin(xyz)$. Find $\delta^2 u / \delta y\, \delta z$.

Ans (7) $e^v = e^{\sin(xyz)}$. $\delta^2 u / \delta y\, \delta z = (\delta/\delta y)(\delta u / \delta z) = (\delta/\delta z)(\delta u / \delta y) = u_{32} = u_{23}$
So, $u = e^{\sin(xyz)}$. $\delta u / \delta y = xz\cos(xyz)e^{\sin(xyz)}$
So, $\delta^2 u / \delta y\, \delta z = \delta[xz\cos(xyz)]e^{\sin(xyz)}] / \delta z$
$= [xz\cos(xyz)][xy\cos(xyz)e^{\sin(xyz)}] + e^{\sin(xyz)}[-xzxy\sin(xyz) + x\cos(xyz)]$ (product rule)
So, $\delta^2 u / \delta y\, \delta z = x^2yz\cos^2(xyz)e^v - x^2yz\sin(xyz)e^v + x\cos(xyz)e^v$.

(8) $u = f(q(x,y), r(x,y))$ $q(x,y) = x^2 - y$; $r(x,y) = x + y^2$
Find $\delta^2 u / \delta x\, \delta y$.

Ans (8) u has functions q and r as variables, q and r are also functions with x and y as variables.
So, $\delta u / \delta x = f_1 q_1 + f_2 r_1$ and $\delta u / \delta y = f_1 q_2 + f_2 r_2$ (chain rule)
So, $\delta^2 u / \delta x\, \delta y = f_1 q_{12} + f_2 r_{12} + q_1(f_{11} q_2 + f_{12} r_2) + r_1(f_{21} q_2 + f_{22} r_2)$
$q_1 = 2x$, $q_{12} = 0$, $q_2 = -1$, $r_1 = 1$, $r_{12} = 0$, $r_2 = 2y$
So, $\delta^2 u / \delta x\, \delta y = f_1(0) + f_2(0) + 2x(f_{11}(-1) + 2yf_{12}) + 1(f_{21}(-1) + 2yf_{22})$
So, $\delta^2 u / \delta x\, \delta y = -2xf_{11} + (4xy - 1)f_{12} + 2yf_{22}$. Note, $f_{12} = f_{21}$.

Advanced Calculus - Partial Derivatives

(9) $u = f(x + u, yu)$. Find $\delta u / \delta x$, $\delta u / \delta y$

Ans (9) $\delta u / \delta x = f_1(1 + \delta u / \delta x) + f_2(y \delta u / \delta y)$
So, $(1-f_1-yf_2) \delta u / \delta y = f_1$. So, $\delta u / \delta x = f_1/(1 - f_1 - yf_2)$.
$\delta u / \delta y = f_1(\delta u / \delta y) + f_2(u + y\delta u / \delta y)$
So, $\delta u / \delta y = uf_2/(1 - f_1 - yf_2)$.

(10) If $\log(uy) + y\log u = x$, Find $\delta y / \delta u$ and $\delta y / \delta x$

Ans (10) partial differentiation with respect to u:
$1/uy(y + u\, \delta y / \delta u) + y(1/u) + (\log u)\delta y / \delta u = 0$
So, $(1/y)\, \delta y / \delta u + (\log u)\delta y / \delta u = -(1/u + y/u) = -(1 + y)/u$
So, $(1/y + \log u)\delta y / \delta u = -(1 + y)/u$
So, $\delta y / \delta u = -[(1 + y)/u]/[(1 + y\log u)/y] = -(y + y^2)/(u + uy\log u)$.
partial differentiation with respect to x:
$(1/uy)(u\delta y / \delta x) + (\log u)\delta y / \delta x = 1$
So, $(1/y)\delta y / \delta x + (\log u)\delta y / \delta x = 1$
So, $\delta y / \delta x = y/(1 + y\log u)$.

(11) If $\sin zy = \cos zx$, Compute $\delta z / \delta x$ when $z = \pi$, $x = 1/3$, $y = 1/6$.

Ans (11) $(\cos zy)(y\, \delta z / \delta x) = -(\sin zx)(x\, \delta z / \delta x + z)$
So, $(y\cos zy + x\sin zx)\, \delta z / \delta x = - z\sin zx$
So, $\delta z / \delta x =- z\sin zx/(y\cos zy + x\sin zx)$
So, $\delta z / \delta x = -\pi\sin(\pi/3) / [(1/6) \cos(\pi/6) + (1/3)\sin(\pi/3)$
So, $\delta z / \delta x = -\pi(\sqrt{3}/2) /[\sqrt{3}/2(1/6 + 1/3)] = - 2 \pi$.

(12) If $\delta u(x,y) / \delta x = 0$, Find $u(x,y)$.

Ans (12) $u(,xy) = g(y)$

(13) If $\delta^2 u(x,y) / \delta x\, \delta y = 0$, Find $u(x,y)$.

Ans (13) $u(x,y) = f(x) + g(y)$.

Peter Oye Simate Sagay
Simate was my mother
Sagay was my father.