Advanced Calculus – Solving PDE By The Method Of Separation Of Variables

/ is the division symbol

It is often desired to find the function that is the basis of a partial differential equation (PDE). The process of finding the function, is *solving the PDE* and the function when found is , the *solution of the PDE*. The method of Separation Of Variables is often used to solve PDEs with the following characteristics:

- The PDE is linear and homogeneous.
- The boundary conditions (BC) are linear and homogeneous.

PDEs with boundary conditions (BC) and initial conditions (IC) are called **Initial Boundary Value Problems** (IBVP). All existential problems have some type of boundaries and starting points. So, a mathematical model that adequately describes existential problems must include the boundary conditions and initial conditions as appropriate for the given problem.

(1) What is the solution to the following Initial Boundary Value Problem (IBVP):

PDE	$u_t = u_{xx}$	$0 < x < 1$
BCs	$u(0, t) = 0$; $u(1, t) = 0$	$0 < t < \infty$
IC	$u(x, 0) = \sin(2\pi x) + (1/3)\sin(4\pi x) + (1/5)\sin(6\pi x)$	

Ans (1) Satisfying the PDE

The IBVP indicates that the function of interest is u(x, t). So, to separate the variables, x and t, we set u(x, t) = X(x)T(t), where X(x) is a function of the independent variable x and T(t) is a function of the independent variable t.

Now, If $u(x, t) = X(x)T(t)$, then $u_t = X(x)T'(t)$ and $u_{xx} = X''(x)T(t)$
Where $T' = \delta (u, t) / \delta t$; $X'' = \delta^2 (u, t) / \delta x\, \delta x$.

So, $X(x)T'(t) = X''(x)T(t)$
So, $T'(t) / T(t) = X'' / X(x) = k = -\lambda^2$
Variables have been separated; k and λ are nonzero constants , $-\lambda^2$ replaces k for convenience
So, $T' + \lambda^2 T(t) = 0$ -------------------------(1)
and $X'' + \lambda^2 X(x) = 0$ -------------------------(2)
Equations (1) and (2) are standard ordinary differential Equations (ODE) and their solutions are:

$T(t) = Ae^{-\lambda^2 t}$ (A an arbitrary constant)
$X(x) = A\sin(\lambda x) + B\cos(\lambda x)$ (A and B are arbitrary constants)
So , $u(x,t) = T(t)X(x) = e^{-\lambda^2 t}[A\sin(\lambda x) + B\cos(\lambda x)]$ -----------------(3)
There are infinite number of u(x,t) of equation (3) that satisfy the PDE, $u_t = u_{xx}$.
So, we reduce this population by seeking u(x,t) that also satisfy the boundary conditions.

Satisfying the PDE and BCs

$u(x,t) = e^{-\lambda^2 t}[A\sin(\lambda x) + B\cos(\lambda x)]$ -----------------(3)
BCs $u(0, t) = 0$; $u(1, t) = 0$ $0 < t < \infty$
So, $u(0,t) = B e^{-\lambda^2 t} = 0$ implies $B = 0$
$u(1,t) = Ae^{-\lambda^2 t}\sin(\lambda) = 0$ implies $\sin(\lambda) = 0$ or $A = 0$. But $A = 0$ implies zero solution.
So to satisfy this BC, it is necessary that $\lambda_n = \pm n\pi$, $n = 1, 2, 3, 4, \ldots$

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So, $u_n(x,t) = A_n e^{-(n\pi)^2 t} \sin(n\pi x)$ --------------------------(4) $n = 1, 2, 3, \ldots$
Equation (4) satisfies both the PDE and Bcs. However there are infinite number of functions, $u_n(x,t)$, called the fundamental solutions, that satisfy (4).

Satisfying the PDE, BCs and IC
It is proposed that u(x,t) = the sum of the fundamental solutions if the coefficients A_n are selected so that IC is satisfied.

So, $u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 t} \sin(n\pi x)$

$IC = u(x,0) = \psi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$

So, $\psi(x) = A_1 \sin(\pi x) + A_2 \sin(2\pi x) + A_3 \sin(3\pi x) \ldots$ -----------------------------(5)
multiply each side of equation (5) by $\sin(m\pi x)$ and integrate from 0 to 1, (m is an arbitrary integer)

So, $\int_0^1 \psi(x) \sin(m\pi x)\, dx = A_m \int_0^1 \sin^2(m\pi x)\, dx = (1/2)A_m$

All other terms reduce to zero by the property of orthogonality:

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\, dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \end{cases}$$

So, $A_m = 2\int_0^1 \psi(x) \sin(m\pi x)\, dx$

So, $u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 t} \sin(n\pi x)$

Where $A_n = 2\int_0^1 \psi(x) \sin(m\pi x)\, dx$

In this particular problem, $IC = u(x,0) = \psi(x) = \sin(2\pi x) + (1/3)\sin(4\pi x) + (1/5)\sin(6\pi x)$
So, if we evaluate A_n, we have $A_2 = 1$, $A_4 = 1/3$, $A_5 = 1/5$. All others zero by orthogonality.
So, $u(x,t) = e^{-(2\pi)^2 t} \sin(2\pi x) + (1/3)e^{-(4\pi)^2 t} \sin(4\pi x) + (1/5)e^{-(6\pi)^2 t} \sin(6\pi x)$.

Peter Oye Simate Sagay
Simate was my mother
Sagay was my father.